

On Top- k Search and Range Reporting

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Abstract

This paper considers top- k search of the following style. Let S be a set of elements, each associated with a real-valued *score*. Let Q be a (possibly infinite) set of predicates. Given a predicate $q \in Q$ and an integer $k \geq 1$, a *top- k query* reports the k elements in $S(q)$ with the highest scores, where $S(q)$ denotes the set of elements in S satisfying q . The objective is to store S in a structure so that queries can be answered efficiently.

We present a general technique that reduces top- k search to three related problems: (i) *counting*: finding the size of $S(q)$, (ii) (essentially) *max-search*: finding the maximum score of the elements in $S(q)$, and (iii) *τ -reporting*: finding all the elements in $S(q)$ whose scores are at least a value τ given at query time. For a number of fundamental top- k problems, their counting, max-search, and τ -reporting variants have already been settled previously. Our technique immediately leads to Las Vegas structures for solving those top- k problems with good expected efficiency.

As a second step, we improve the query performance for the *top- k range reporting* problem. Specifically, the elements in S are one-dimensional points. Given a range $q = [x_1, x_2]$ and an integer $k \geq 1$, a query returns the k points in $S \cap q$ with the highest scores. In external memory, we give a linear-size Las Vegas structure that answers a query in $O(\lg_B n + k/B)$ I/Os and can be updated in $O(\lg_B n)$ amortized I/Os per insertion and deletion, where $n = |S|$ and B is the block size. The space cost is in the worst case, whereas the query and update time holds with high probability and in expectation simultaneously. To obtain this result, we develop a technique that combines random sampling (for handling large k) and tabulation (for small k), which we believe is of independent interest.

1 Introduction

In a typical reporting problem, a query returns all the input elements satisfying a certain predicate. In some applications, elements have priorities such that only the most important ones should be reported. For instance, a user looking for a hotel may request “the k best-rated hotels whose prices are between 100 and 200 dollars per night”, where k depends on her/his preference and is given only at run time. The request can be entertained by first finding all the hotels fulfilling the price condition, and then fetching the best k . Apparently, this approach has the drawback of incurring long response time when many hotels pass the price check. Ideally, we should be able to find the top- k hotels directly without wasting time extracting the rest. Similar situations have been described in many different areas, including database [27], networking [21], web search [25], to mention just a few.

Motivated by this, we study *top- k search* of the following style. Let S be a set of elements, each associated with a distinct real-valued *score*. A query is given a predicate q drawn from a (perhaps infinite) set Q , and an integer $k \geq 1$. It reports the k elements in $S(q)$ with the highest scores, where $S(q)$ denotes the set of elements in S satisfying q . If $|S(q)| < k$, the entire $S(q)$ is returned. The objective is to store S in a structure so that queries can be answered efficiently.

Models and conventions. Our discussion will assume the *external memory model* (a.k.a. the I/O model) [4]. Let B be the size (in words) of a disk block. A word is assumed to have $\Omega(\lg n)$ bits, where n is the underlying problem’s input size. Space is measured by the number of blocks occupied by a structure, whereas time is by the number of I/Os performed by an algorithm. Compared to word-RAM, the external memory model as the “advantage” of free CPU calculation. We will not abuse the advantage, so that all our results hold directly in word-RAM with B set to an appropriate constant.

A complexity holds in the worst case unless otherwise stated. A logarithm $\lg_b x$ is defined as $\max\{1, \log_b x\}$, and has base $b = 2$ by default. The input size n is at least B . Linear cost is understood as $O(n/B)$ whereas logarithmic cost is $O(\lg_B n)$. *With high probability* (henceforth, w.h.p.) refers to a probability at least $1 - 1/n^2$, where the exponent 2 can be replaced with any positive constant. All lemmas and theorems hold when n is greater than a sufficiently large constant.

1.1 Our results

Top- k search has some natural companion problems, which assume the same input S and predicate set Q , but differ in query formulation. In the first companion, the *counting* problem, a query is given a predicate $q \in Q$, and returns $|S(q)|$. In the second, the *top-constant-score* problem, a query is given $q \in Q$, and returns the c -th highest score of the elements in $S(q)$, where c is a constant fixed for *all* queries. If $|S(q)| < c$, the query returns ∞ . In the third companion, the *τ -reporting* problem, a query is given $q \in Q$ and a real value τ , and reports the elements in $S(q)$ with scores at least τ .

We show that top- k search is a combination of the above three companions in disguise, as far as *expected* efficiency is concerned. Specifically, we give a general technique to reduce top- k search to its companions with no performance deterioration, i.e., the (expected) space, query and update cost of the resulting top- k structure is determined by the most expensive of the companion structures. The technique gives rise to elegant Las Vegas structures for the top- k version of several fundamental problems:

Top- k range reporting. In this problem, S is a set of points in the real domain \mathbb{R} . Given an interval $q = [x_1, x_2]$ in \mathbb{R} and an integer k , a query reports the k points with the highest scores in $S(q) = S \cap q$. Our structure uses linear space (expected and w.h.p.), answers a query in $O(\lg_B n + k/B)$ expected I/Os, and supports an insertion and deletion in $O(\lg_B n)$ amortized I/Os (expected and w.h.p.). This is the first structure to achieve logarithmic query and update time in external memory (similar results

were known in internal memory nearly two decades ago [13]). Previously, the best structure [27] has linear space and $O(\lg_B n + k/B)$ query cost, but requires $O(\lg_B^2 n)$ amortized I/Os to handle an update.

Top- k stabbing. S is a set of intervals in \mathbb{R} . Given a value $q \in \mathbb{R}$ and an integer k , a query reports the k intervals with the highest scores among the intervals of S containing q . Our structure uses linear space (expected and w.h.p.), answers a query in $O(\lg_B n + k/B)$ expected I/Os, and supports an insertion and deletion in $O(\lg_B n)$ amortized I/Os (expected and w.h.p.). We are not aware of any previous result on this problem. For $k = 1$, the problem is known as *stabbing max*, for which Agarwal et al. [3] presented a linear size structure with logarithmic query and amortized update time.

Top- k 3-sided range search. S is a set of points in \mathbb{R}^2 . Given a rectangle $q = [x_1, x_2] \times [y, \infty)$ and an integer k , a query reports the k points with the highest scores in $S \cap q$. We obtain a static structure that uses $O(\frac{n}{B} \lg \frac{n}{\lg_B n})$ space (expected and w.h.p.) and answers a query in $O(\lg_B n + k/B)$ expected I/Os. The space-query tradeoff matches the best achievable by any deterministic structure. We again are not aware of any previous result.

It is perhaps remarkable how effortlessly we develop the structures from our reduction, because none of the three problems appears to be trivial at all. By contrast, the previous structures [2, 27] for top- k range reporting are much more involved.

The query efficiency of our reduction does not necessarily hold w.h.p. To make that happen, special effort is needed to hack into the black box. In this paper, we demonstrate so for top- k range reporting: our ultimate structure uses $O(n/B)$ space (worst case), answers a query in $O(\lg_B n)$ I/Os (expected and w.h.p.), and can be updated in $O(\lg_B n)$ amortized I/Os (expected and w.h.p.). For $k = O(\text{poly} \lg n)$, the update and query time holds in the worst case.

1.2 Techniques

Given elements e_1 and e_2 from an ordered domain, we say that e_1 is *smaller* than e_2 if e_1 precedes e_2 ; otherwise, e_1 is *greater* than e_2 . Let L be a set of elements from an ordered domain. Given an element e , define its *rank* in L as the number of elements in L at least e (i.e., the largest element of L has rank 1). This work initiated from a rudimentary observation. Take a sample set R of L by including each element independently with probability p . Suppose that we want to find the element with rank k in L . Then, the element with rank roughly kp in R should be quite close to what we are looking for.

The proposed reduction results from setting kp to an appropriate constant in the above “rank sampling” observation. This enables us to retrieve just the top *few* elements in R , and use their scores to search for the real top- k elements. The challenge is to account for the error of sampling. We show how to keep the error in check by preparing multiple sample sets with exponentially decaying sampling rates, and a doubling trick in query processing.

We now turn attention to ensuring good query efficiency w.h.p. for top- k range reporting. Unfortunately, rank sampling falls short for this purpose: its reliance on statistical significance prevents it from working on small k . Following [27], we instead target a different problem:

approximate range k -selection. Let S be as defined in top- k range reporting. Given an interval $q = [x_1, x_2]$ in \mathbb{R} and an integer k satisfying $1 \leq k \leq |S \cap q|$, a query returns a point $e \in S \cap q$ such that at least k but less than ck points in $S \cap q$ have higher scores than e , where $c > 1$ is a constant fixed for all queries.

Suppose that a linear-size structure solves the above problem under an arbitrary c with query time t_q and update time t_u . This directly implies a linear-size structure for top- k range reporting with query time $t_q + O(\lg_B n + k/B)$ and update time $t_u + O(\lg_B n)$ [27]. Indeed, the main contribution of [27] is a structure with $t_q = O(\lg_B n)$ and $t_u = O(\lg_B^2 n)$. Our goal is to lower t_u to $O(\lg_B n)$.

Rank sampling again lends a helping hand: it allows us to exploit an $O(\text{polylg } n)$ -update time structure to handle queries with large k . Intuitively, we can set the sampling rate p to roughly $1/\text{polylg } n$, and then apply the structure on the sample set. Since the number of updates has been decreased by $\text{polylg } n$ times, the update cost is brought down to $O(1)$ amortized. Combining the idea with the structure of [27], we achieve query time $O(\lg_B n)$ and update time $O(\lg_B n)$ w.h.p. for $k = \Omega(\lg^2 n)$.

The sampling technique no longer works for $k = O(\lg^2 n)$, whose handling eventually boils down to another problem:

approximate (f, l) -group k -selection. An (f, l) -group G is a list of f disjoint sets G_1, \dots, G_f , each of which has at most l elements drawn from the same ordered domain. Given an interval $q = [\alpha_1, \alpha_2]$ with $1 \leq \alpha_1 \leq \alpha_2 \leq f$ and an integer k with $1 \leq k \leq |\bigcup_{i \in q} G_i|$, a query returns an element whose rank in $\bigcup_{i \in q} G_i$ falls in $[k, ck)$, where $c > 1$ is a constant fixed for all queries.

The difficulty is to solve the problem when $f = \lg^\epsilon n$ for some positive $\epsilon < 1$, $l = O(\text{polylg } n)$, and $\lg \lg n \geq \sqrt{B}$. We observe that under those constraints, the problem can be dealt with using tabulation. Specifically, we precompute a table of $o(n)$ bits so that any instance of the problem can be represented in a compressed form that guarantees query time $O(\lg_B(fl))$. Furthermore, the table even captures all the *transitions* from one compressed form to another due to an update in any of the sets in G . In this way, we support an update in $O(\lg_B(fl))$ I/Os, which in turn gives a desired deterministic structure for approximate range k -selection with $k = O(\text{polylg } n)$. Our approach abandons the *indivisibility assumption*, and demonstrates new power in fully leveraging all the bits available for storage. Even though eliminating indivisibility has prevailed the word-RAM model, research of this form has emerged only recently in external memory [19, 22].

At a high level, our solution presents a framework for designing top- k structures with logarithmic update time w.h.p. First, obtain a structure that is query efficient but incurs $O(\text{polylg } n)$ update time. Then, applying rank sampling, we use this structure to handle large $k = \Omega(\text{polylg } n)$, and tackle the case $k = O(\text{polylg } n)$ by combining approximate rank selection with our compression technique. We believe that this framework is of independent interest.

Remark. Using random sampling to approximate ranks is not new. This was a main idea behind many classic results whose complete coverage is far beyond this paper (see, e.g., [7, 10, 11, 20]). What remains non-trivial is how to employ it with other observations to obtain an elegant reduction for generic top- k search. Moreover, how we combine sampling with tabulation in designing external memory structures is also new to our best knowledge.

1.3 Previous results

Top- k range reporting was first studied by Afshani, Brodal and Zeh [2], who gave a static linear-size structure with $O(\lg_B n + k/B)$ query time. They also analyzed the space-query tradeoff for an *ordered* variant of the problem, where the top- k elements need to be reported in the order of their scores (no order requirement exists in our problems). As pointed out in [27], the lower bound of [2] suggests that, when space usage must be linear, a simple approach already achieves near-optimal query efficiency: first solve the unordered version in $O(\lg_B n + k/B)$ I/Os, and then sort the result elements. For the unordered version, Cheng and

Tao [27] proposed a dynamic structure that matches the space and query cost of [2], but can also be updated in $O(\lg_B^2 n)$ amortized I/Os per insertion and deletion.

In RAM, by combining a priority search tree [23] and Frederickson's selection algorithm [13] on min-heaps, one can obtain a structure that uses $O(n)$ words, answers a query in $O(\lg n + k)$ time, and supports an insertion and a deletion in $O(\lg n)$ time. It is unclear whether the structure can be adapted to work efficiently in external memory (straightforward adaptation results in $O(\lg n + k)$ query time, instead of $O(\lg_B n + k/B)$). Brodal, Fagerberg, Greve and Lopez-Ortiz [9] considered a special instance of the problem where the input points of S are from the domain $[n]$.¹ They gave a linear-size structure with $O(1 + k)$ query time (which holds also for the ordered version). Note that, by fixing $k = 1$, the problem studied in [9] specializes into the well-known *range minimization query* problem [17].

We are not aware of existing results for top- k stabbing and 3-sided range search. In [22], Larsen and Pagh mentioned an application of top- k 3-sided range search: a solution to this problem also settles a document selection problem called *top- k colored prefix reporting*. They presented a structure in the *scatter I/O model*, where an I/O may read/write B arbitrary words in the disk (i.e., these words need not be consecutive). Our top- k 3-sided structure is the first one in the traditional external memory model solving the top- k colored prefix reporting problem with logarithmic query cost plus linear output time.

The exact version of approximate range k -selection (i.e., a query should return the point with precisely the specified rank) has been studied in internal memory as the *range median* problem. Currently, the best dynamic structure [12] uses $O(n)$ space, answers a query and supports an update in $O((\frac{\lg n}{\lg \lg n})^2)$ time. This structure does not suit our needs because we aim at logarithmic query and update time, not to mention that this structure remains to be externalized.

2 A reduction technique for general top- k search

This section will describe a framework of designing index structures for top- k search with attractive expected efficiency. We will then apply the framework to solve the three instances of top- k search mentioned in Section 1.1.

2.1 Rank sampling

Let L be a set of elements. We say that R is a *p-sample set* of L if R is obtained by independently including each element of L with probability p .

Lemma 1. *Let R be a p -sample set of L . Let $k \geq 1$ and $\delta \in (0, 1)$ satisfy $kp \geq 3 \ln \frac{3}{\delta}$. If $|L| \geq 4k$, the following hold simultaneously with probability at least $1 - \delta$:*

- $|R| \geq 2kp$, and
- the element with rank $\lceil 2kp \rceil$ in R has rank between k and $4k$ in L .

All the omitted proofs (such as the above one) can be found in the appendix.

2.2 Reduction

We now present a reduction from top- k search to counting, top-constant-score search, and τ -reporting. Let S be the input set of the top- k problem, and $n = |S|$. For each $i \in [0, \lfloor \lg n \rfloor]$, take a p_i -sample set S_i of S , where $p_i = \min\{1, \frac{1}{2^i} \cdot 3 \ln 9\}$. Index each S_i with a structure for answering top- $\lceil 6 \ln 9 \rceil$ -score queries.

¹Given an integer $x \geq 1$, $[x]$ represents the set of integers between 0 and $x - 1$.

Finally, store S in two structures for answering count and τ -reporting queries, respectively. The τ -reporting structure needs to be worst-case query efficient. Let $T_{rep} + c \cdot t/B$ be the worst-case time required to answer a τ -reporting query on a set of n elements, where $c > 0$ is a constant, and t is the number of elements reported.

Let us see how to answer a top- k query with predicate $q \in Q$. If $k \leq 3 \ln 9$, run a top- k -score query on $S_0 (= S)$ to obtain a score s , and then report the result of an s -reporting query on S . If $n/2 \leq k \leq n$, scan the entire S to retrieve $S(q)$, i.e., the set of elements in S qualifying q . Then, perform k -selection (in linear time [8]) on the elements in $S(q)$ to find the k -th highest score s of the elements there, and report all elements in $S(q)$ whose scores are at least s .

For $k \leq n/2$, the query algorithm first runs a count query to obtain $|S(q)|$. The rest of the algorithm executes in *rounds* with a parameter k' , where k' is set to the smallest power of 2 at least k in the first round, and doubles each time a new round begins. In each round, if $|S(q)| < 4k'$, retrieve $S(q)$ with an ∞ -reporting query on S with predicate q , find the top- k elements from $S(q)$ by k -selection, and finish the entire algorithm. Otherwise, perform a top- $\lceil 6 \ln 9 \rceil$ -score query on $S_{\lg k'}$; let s be the score returned by this query. Run an s -reporting query on S with predicate q in a *cost-monitoring* manner: force the query to terminate as soon as its cost reaches $T_{rep} + c \cdot 4k'/B$ – if the query has not terminated normally at this point, we say that it has *aborted*. If the query terminated normally (i.e., it did not abort), let S' be the set of elements retrieved. If $|S'| \geq k$, finish the algorithm by finding the top- k elements in S' with k -selection. If $|S'| < k$ or the s -reporting query aborted, double k' , and perform another round with the new k' .

If the counting, top- $\lceil 6 \ln 9 \rceil$ -score and τ -reporting structures all support updates, they can be maintained in a straightforward manner along with the updates on S , so that our overall top- k structure is dynamic. In particular, an update concerns the top- $\lceil 6 \ln 9 \rceil$ -score structure on S_i if and only if the element being inserted/deleted is sampled in S_i . For each element of S , we can easily keep track of the sample sets it belongs to with hashing.

Next, we analyze the performance guarantees of our reduction. For this purpose, suppose that, on a set of n elements, a count, top- $\lceil 6 \ln 9 \rceil$ -score, and τ -reporting query can be answered in T_{cnt} , T_{topc} , and (as mentioned before) $T_{rep} + O(t/B)$ time, respectively. Furthermore, suppose that the structures for those queries can be updated in U_{cnt} , U_{topc} and U_{rep} amortized time, respectively (if a structure is static, its update time is ∞). We have:

Theorem 1. *The structure obtained with our reduction correctly answers every top- k query with $T_{cnt} + O(T_{topc} + T_{rep} + k/B)$ I/Os in expectation. It supports any sequence of n updates on an initially empty input with $n(U_{cnt} + U_{rep} + O(U_{topc}))$ I/Os both in expectation and w.h.p.*

Proof. We prove only the query cost, and leave the update time to the appendix. It is obvious that: (i) for $k \leq 3 \ln 9$, the query cost is $T_{topc} + T_{rep} + O(1)$, and (ii) for $k > n/2$, the query cost is $O(n/B) = O(k/B)$. Next, we assume $3 \ln 9 < k \leq n/2$. Set $c = \lceil 6 \ln 9 \rceil$ in the rest of the proof.

Consider the execution of a round with parameter k' . If $|S(q)| < 4k'$, the round performs $T_{rep} + O(k'/B)$ I/Os. Otherwise, we apply Lemma 1 with $\delta = 1/3$: since $k' p_{\lg k'} = \min\{k', k' \frac{3 \ln 9}{2^{\lg k'}}\} = \min\{k', 3 \ln 9\} > 3 \ln \frac{3}{\delta}$, we know that with probability at least $2/3$, (i) S_i has at least $\lceil 2k' p_{\lg k'} \rceil \geq c$ elements qualifying q , and (ii) the element in S_i with the c -th largest score has score rank between k' and $4k'$ in $S(q)$. When both conditions hold, the algorithm issues a τ -reporting query that returns at most $4k'$ elements, and terminates after k -selection. In this case, the cost of this round is at most $T_{topc} + T_{rep} + O(k'/B)$. The algorithm does not finish at this round with probability at most $1/3$ due to two possibilities: the τ -reporting query on S fetched less than k' objects, or this query aborted. For either possibility, this round performs at most $T_{topc} + T_{rep} + O(k'/B)$ I/Os.

In summary, in the i -th ($i \geq 1$) round, the algorithm terminates with probability at least $2/3$, and goes into the next round with the remaining probability. In any case, a round performs $T_{topc} + T_{rep} + O(2^i \cdot k/B)$ I/Os ($k \leq k' < 2^i \cdot k$ in the i -th round). Due to the independence of all the sample sets, the i -th round is executed with probability at most $(1/3)^{i-1}$. The expected query cost is therefore at most

$$T_{cnt} + \sum_{i=1}^{\infty} (1/3)^{i-1} (T_{topc} + T_{rep} + O(2^i \cdot k/B)) = T_{cnt} + O(T_{topc} + T_{rep} + k/B).$$

It can be easily verified that the query algorithm is Las Vegas. \square

Remark. Top-constant-score queries are seldom specifically studied in the literature. A related, more popular, topic is the *max-score* problem, namely, top-1-score search. Extending a max-score structure into a top-constant-score one is often effortless. In particular, if the max-score structure supports fast updates, a simple way of answering a top- c -score query is to repeat the following c times: issue a max-score query, retrieve the element of the maximum score (recall that all elements have distinct scores), and delete the element from the structure. After the query, insert the c deleted elements back into the structure.

2.3 Applications

We now apply Theorem 1 to solve the top- k problems mentioned in Section 1.1:

Corollary 1. *For top- k range reporting, there is a structure that uses linear space (expected and w.h.p.), answers a query in $O(\lg_B n + k/B)$ expected I/Os, and can be updated in $O(\lg_B n)$ amortized I/Os (expected and w.h.p.). The same is true for top- k stabbing. For top- k 3-sided range search, there is a structure using $O(\frac{n}{B} \frac{\lg n}{\lg \lg_B n})$ space (expected and w.h.p.), and answering a query in $O(\lg_B n + k/B)$ expected I/Os.*

Our (randomized) structure for top- k 3-sided range search achieves the best space-query tradeoff one can hope for with any deterministic structure.

Theorem 2. *Under the indexability model of [18], any deterministic structure for top- k 3-sided range search must use $\Omega(\frac{n}{B} \frac{\lg n}{\lg \lg_B n})$ space if it guarantees query time $O(\lg_B^c n + k/B)$ for any constant c .*

3 Improved structure for top- k range reporting

In this section, we present a better structure for top- k range reporting, which achieves $O(\lg_B n + k/B)$ query time w.h.p. Recall that the input S is a set of points in \mathbb{R} , each of which is associated with a score in \mathbb{R} . Given an interval $q = [x_1, x_2]$ in \mathbb{R} and an integer $k \geq 1$, a query returns the k elements in $S(q) = S \cap q$ with the highest scores. We will assume $|S(q)| \geq 4k$ because otherwise we can retrieve the entire $S(q)$ from a B-tree on S , and apply k -selection. Unless otherwise stated, in a B-tree, a leaf node stores at most B elements, and an internal node has at most B child nodes.

3.1 Handling $k = \Omega(\lg n \cdot \lg_B n)$

Maintain a p -sample set R of the input set S , where $p = 1/\lg_B n$. Store R in an approximate range k -selection structure T_{sel} of [27]. Furthermore, index S and R respectively with an augmented B-tree for answering count queries, i.e., given $q = [x_1, x_2]$, return $|S(q)|$ and $|R(q)|$, where $R(q) = R \cap q$. Index S with another B-tree that allows us to retrieve efficiently all the points in $S(q)$, and also an external priority search tree T_{prior} of [5] to support τ -reporting queries on S .

Given a top- k query with $q = [x_1, x_2]$ and $k \geq 7 \ln n \cdot \lg_B n$, first check whether $|R(q)| \geq 2kp$. If yes, perform approximate rank k' -selection with $q = [x_1, x_2]$ and $k' = \lceil 2kp \rceil$ on T_{sel} ; let s be the score of the retrieved element. Otherwise (i.e., $|R(q)| < 2kp$), set $s = \infty$ directly. Use T_{prior} to retrieve all the points of $S \cap q$ whose scores are at least s . If at least k elements are fetched, find the top- k elements with k -selection. Otherwise (i.e., less than k elements are fetched), retrieve the entire $S(q)$ and find the top- k elements with k -selection.

Lemma 2. *For top- k range reporting with $k \geq 7 \ln n \cdot \lg_B n$, there is a structure that uses $O(n/B)$ space in the worst case, correctly answers every query with $O(\lg_B n + k/B)$ I/Os in expectation and w.h.p., and supports n updates on an initially empty input with totally $O(n \lg_B n)$ I/Os in expectation and w.h.p.*

3.2 The approximate (f, l) -group k -selection problem

In this section, we give a dynamic structure solving the approximate (f, l) -group k -selection problem (henceforth, the (f, l) -problem) defined in Section 1.2 when $f = \lg^\epsilon n$, $l = O(\text{poly} \lg n)$, and $\lg \lg n \geq \sqrt{B}$. Abusing notation slightly, we use G also to denote the union of G_1, \dots, G_f .

Logarithmic sketch. We first review the *logarithmic sketch* (henceforth, *sketch*) developed in [27]. Let L be a set of l ordered elements. Its sketch Σ is an array of size $\lfloor \log_2 l \rfloor + 1$, where the j -th ($j \geq 1$) entry $\Sigma[j]$ is an element in L whose rank in L falls in $[2^{j-1}, 2^j)$ (any such element can be used for $\Sigma[j]$). We refer to $\Sigma[j]$ as a *pivot*. The following is from [27]:

Lemma 3 ([27]). *Let L_1, \dots, L_m be m disjoint sets of elements drawn from an ordered domain. Given their sketches and an integer k satisfying $1 \leq k \leq |\bigcup_{i=1}^m L_i|$, we can find in $O(m)$ I/Os an element $e \in \bigcup_{i=1}^m L_i$ whose rank in $\bigcup_{i=1}^m L_i$ is at least k but less than ck for some constant $c > 1$.*

Static structure. We now describe a static structure for the (f, l) -problem. Create a sketch Σ_i for each G_i ($1 \leq i \leq f$). We force Σ_i to have $\lfloor \log_2 l \rfloor + 1$ pivots: if $|G_i|$ is less than l , the last few pivots in Σ_i are *dummy*. Call the set $\{\Sigma_1, \dots, \Sigma_f\}$ a *sketch set*.

We store a compressed form of the sketch set as follows. We describe each pivot $e \in \Sigma_i$ by its *global rank* in G using $\lg(fl)$ bits. Also, we associate e with its *local rank* in G_i , which can be described in $\lg l$ bits. We use the same number $O(\lg(fl))$ of bits for each pivot, even for a dummy one. Hence, each Σ_i occupies $O(\lg l \cdot \lg(fl))$ bits; and thus, a compressed sketch set occupies $O(f \lg l \cdot \lg(fl)) = O(\lg^\epsilon n \cdot (\lg \lg n)^2)$ bits – note that this is less than the length of a word. There are at most $h = 2^{O(\lg^\epsilon n \cdot (\lg \lg n)^2)}$ possible compressed sketch sets.

Besides an integer $k \in [1, fl]$, a query for the (f, l) -problem specifies a range $[\alpha_1, \alpha_2]$ such that $1 \leq \alpha_1 \leq \alpha_2 \leq f$. Hence, there are less than $f^2 \cdot fl = f^3 l$ possible queries, each of which can be described in $O(\lg(fl))$ bits. For each possible combination of (compressed sketch set, query), we compute in $O(f)$ I/Os the query answer (using Lemma 3), which is an element in G , and described by its global rank in $O(\lg(fl))$ bits. Create a *query lookup table* \mathbb{T}_{qry} where each entry corresponds to a (compressed sketch set, query) pair, and stores the query answer in its global rank. All entries are arranged in the lexicographic order of the bit-description – referred to as a *bit-string* – of the corresponding (compressed sketch set, query) pair to allow constant-time lookups. \mathbb{T}_{qry} occupies

$$O(h \cdot f^3 l \cdot \lg(fl)) = 2^{O(\lg^\epsilon n \cdot (\lg \lg n)^2)} \cdot O(\text{poly} \lg n) = o(n)$$

bits, and can be computed in $O(h \cdot f^3 l \cdot f) = O(h \cdot \text{poly} \lg n) = o(n)$ I/Os.

Given a query, we append its bit-string to that of the current compressed sketch set to acquire the corresponding entry of \mathbb{T}_{qry} in constant time. Remember that the query answer is given in global rank. We

therefore index all the elements of G with a B-tree, so that we can convert a global rank to an actual element in $O(\lg_B(fl))$ I/Os. The B-tree occupies $O(fl/B)$ space².

Update. Now we make our structure dynamic. First, create a B-tree on the elements of each G_i , and another B-tree on the (uncompressed) pivots of each Σ_i . Unlike a compressed sketch, the B-tree on Σ_i does *not* contain the global and local ranks of the pivots. At all times, we let each pivot in (the B-tree of) Σ_i and its copy in (the B-tree of) G_i keep a pointer to each other. Later, we need to scan the elements of G_i between two pivots. Those pointers allow us to do so in $O(\lg_B \lg l + t/B)$ I/Os if t elements are scanned. During a node split/merge in a relevant B-tree, such pointers are properly maintained using $O(B)$ I/Os. As $\Omega(B)$ updates must have taken place in the node being split/merged, the $O(B)$ cost can be amortized so that an update bears only $O(1)$ I/Os. In the sequel, we sometimes need to add or remove the *last* pivot of some Σ_i . This can be done in the compressed form of Σ_i and the B-tree on Σ_i in $O(1)$ and $O(\lg_B \lg l)$ I/Os, respectively.

Suppose that an element e_{new} is to be inserted in G_i for some $i \in [1, f]$. Let r_{new} be the rank of e_{new} in G before the insertion. Except perhaps a single pivot, the new compressed sketch set (after the update) can be deduced from: the current compressed sketch set, r_{new} and i . To understand this, consider first a compressed sketch $\Sigma_{i'}$ where $i' \neq i$. Each pivot whose global rank is at least r_{new} now has its global rank increased by 1 (its local rank is unaffected). Regarding the compressed Σ_i , the same is true, but additionally every such pivot should also have its local rank increased by 1. Furthermore, a new pivot is needed in Σ_i if $|G_i|$ reaches a power of 2 after the insertion – in such a case we say that Σ_i *expands*; the new pivot is the only one in the compressed sketch set that cannot be deduced (because its global rank is unknown).

Motivated by this observation, we precompute another table \mathbb{T}_{ins} called the *insertion lookup table*. Let Σ denote a possible compressed sketch set. \mathbb{T}_{ins} has an entry for every possible combination of (Σ, r_{new}, i) , where r_{new} and i are as explained before. This entry contains the new compressed sketch set determined by (Σ, r_{new}, i) , *excluding* the new pivot if Σ_i needs to expand. \mathbb{T}_{ins} has $h \cdot fl \cdot f$ entries, whereas each entry occupies $O(\lg^\epsilon n \cdot (\lg \lg n)^2)$ bits, i.e., the length of a compressed sketch set. We store the entries in the lexicographic order of the bit-strings of the corresponding (Σ, r_{new}, i) to allow constant-time lookups. Overall, \mathbb{T}_{ins} occupies $O(h \cdot \text{poly} \lg n) = o(n)$ bits.

Recall that, in a logarithmic sketch, the j -th pivot should have its local rank confined to $[2^{j-1}, 2^j)$. The pivot is *invalidated* when its local rank equals $2^{j-1} - 1$ or 2^j . When this happens, as in [27], we re-compute the pivot to be the element with local rank $\lfloor \frac{3}{2} \cdot 2^{j-1} \rfloor$ so that $\Omega(2^j)$ updates in G_i are needed for the new pivot to be invalidated. The only exception is when the pivot has local rank $2^{j-1} - 1$ *and* is the last one in its sketch set, but this situation happens only in deletion, which will be discussed later. Pivot re-computation must be done *online* because we cannot predict the new pivot's global rank in advance. Therefore, in each entry of \mathbb{T}_{ins} , we associate the new compressed sketch set Σ' stored there with a pointer to a linked list, which indicates all the invalidated pivots in Σ' . As each pivot can be described by two $O(\lg(fl))$ -bit integers (i.e. which Σ_i contains it, and its rank in Σ_i), all the linked lists occupy $O((h \cdot fl \cdot f) \cdot (fl \cdot \lg(fl))) = o(n)$ bits. For each (Σ, r_{new}, i) , we can easily compute Σ' , as well as all the invalidated pivots in Σ' , in $O(\text{poly} \lg n)$ time, implying that the \mathbb{T}_{ins} can be generated in $o(n)$ I/Os.

We are ready to clarify the full procedure of inserting e_{new} in G_i . After obtaining r_{new} from the B-tree of G in $O(\lg_B(fl))$ I/Os, we lookup \mathbb{T}_{ins} in constant I/Os (note: (Σ, r_{new}, i) can be described by $o(\lg n)$

²If one aims at designing only an external memory structure, the query lookup table is unnecessary. We mentioned earlier that a sketch set can be stored within a single word, which can be loaded into memory with one I/O. Then, we can apply Lemma 3 to answer a query in memory, for which purpose the algorithm of [27] requires only $O(1)$ words of memory, and incurs no cost in our context (recall that CPU time is for free). This approach, however, does not work in word-RAM, where a penalty of $O(f)$ time applies to each query. The above tabulation approach, on the other hand, allows us to handle a query in $O(\lg(fl))$ time even in word-RAM.

bits) to obtain the new compressed sketch set, as well as the list of invalidated pivots, if any. If now $|G_i|$ is a power of 2, we retrieve the global rank of the smallest element $e \in G_i$ in $O(\lg_B(fl))$ I/Os, and add e to Σ_i in $O(\lg_B \lg l)$ I/Os.

Next, we deal with the invalidated pivots. For each such pivot (suppose that it is the j -th pivot of Σ_i), we retrieve the element $e \in G_i$ with local rank $\lfloor \frac{3}{2} \cdot 2^{j-1} \rfloor$ in $O(\lg_B \lg l + 2^j/B)$ I/Os. This can be done by scanning the elements of G_i between $\Sigma_i[j]$ and its succeeding or preceding pivot in Σ_i (if the local rank of $\Sigma_i[j]$ is currently $2^{j-1} - 1$ or 2^j , respectively). To acquire the global rank of e , we distinguish two cases:

- $2^j \geq B \lg_B(fl)$: there are $O(\lg l)$ such j (i.e., the number of pivots in a sketch). We obtain the global rank by searching the B-tree on G in $O(\lg_B(fl))$ I/Os, and then update Σ_i accordingly in $O(\lg_B \lg l)$ I/Os. In total, we spend $O(\lg_B(fl) + 2^j/B) = O(2^j/B)$ I/Os to handle this invalidated pivot. Since as mentioned earlier $\Omega(2^j)$ updates must have occurred in G_i to trigger the invalidity of $\Sigma_i[j]$, each of those updates accounts for $O(1/B)$ I/Os of the invalidation handling. As an update can be charged at most $O(\lg l)$ times this way (i.e., once for every j), its amortized cost is increased by only $O(\frac{1}{B} \lg l) = O(\lg_B l)$.
- $2^j < B \lg_B(fl)$: we retrieve the global rank of e in constant time using:

Lemma 4. *In $o(n)$ I/Os, we can precompute several tables of $o(n)$ bits in total. Given an instance of the (f, l) -problem, there is a structure consuming $O(fl/B)$ space such that, given any integer $r \in [1, B \lg_B(fl)]$ and $i \in [1, f]$, the global rank of the element with rank r in G_i can be retrieved in $O(1)$ I/Os. The structure can be updated in $O(\lg_B(fl))$ I/Os per insertion and deletion. Both the query and update algorithms need to consult the precomputed tables. The structure can be built in $O(fl \cdot \lg_B(fl))$ I/Os.*

Now we can update Σ_i by using e to replace the original $\Sigma_i[j]$ in $O(\lg_B \lg l)$ I/Os. In total, we spend $O(\lg_B \lg l + 2^j/B)$ I/Os to handle the invalidation of this pivot. We only need to worry about the term $O(\lg_B \lg l)$ because the other term $O(2^j/B)$ can be accounted for in the way explained earlier. Note that there are $O(\lg(B \lg_B(fl)))$ values of j satisfying $2^j < B \lg_B(fl)$. In other words, an insertion can trigger the invalidation of $O(\lg(B \lg_B(fl)))$ such pivots, meaning that the insertion cost is increased by $O(\lg(B \lg_B(fl)) \cdot \lg_B \lg l)$. Fitting the values of f and l , we know $O(\lg(B \lg_B(fl))) = O(\lg B + \lg \lg_B(fl)) = O(\lg \lg \lg n + \lg \lg_B \lg n) = O(\lg \lg_B \lg n)$ I/Os, where the last equality used $O(\lg \lg \lg n) = \Theta(\lg \lg_B \lg n)$ when $\sqrt{B} \leq \lg \lg n$. Hence, $O(\lg(B \lg_B(fl)) \cdot \lg_B \lg l) = O(\lg \lg_B \lg n \cdot \lg_B \lg \lg n) = O((\lg \lg_B \lg n)^2) = o(\lg_B(fl))$.

Therefore, an insertion can be performed in $O(\lg_B(fl))$ amortized I/Os. A similar technique handles a deletion with the same cost (see the proof of the next lemma). Combining our earlier discussion on query processing, we have obtained:

Lemma 5. *In $o(n)$ I/Os, we can precompute several tables of $o(n)$ bits in total. Given an instance of the (f, l) -problem, there is a structure consuming $O(fl/B)$ space that (using also the precomputed tables) answers a query in $O(\lg_B(fl))$ I/Os, and can be updated in $O(\lg_B(fl))$ I/Os per insertion and deletion. The structure can be built in $O(fl \cdot \lg_B(fl))$ I/Os.*

We remark that it is unnecessary to improve the $o(n)$ precomputation time to $O(n/B)$. This is because we can re-compute the relevant tables whenever n has changed by a constant factor. The amortized cost per update is only $o(1)$, as discussed in detail later.

3.3 Approximate union-rank selection

This subsection considers the following *approximate union-rank* problem. Let L_1, \dots, L_m be m disjoint sets, whose elements are drawn from an ordered domain. An algorithm is allowed only two operators to access L_i ($1 \leq i \leq m$):

- *max-operator*, which fetches the largest element of L_i in cost_{\max} time.
- *rank-operator*, which takes a real-valued parameter $r \in [1, |L_i|/c]$, and returns an element whose rank in L_i falls in $[r, cr)$, where $c \geq 2$ is a constant. The operator takes $\text{cost}_{\text{rank}}$ time.

Given an integer k with $1 \leq k \leq \frac{1}{c^2} \min\{|L_1|, \dots, |L_m|\}$, the algorithm should return an element whose rank in $L_1 \cup \dots \cup L_m$ falls in $[k, c'k)$, where $c' > 1$ is a constant dependent only on c .

The classic algorithm of Frederickson and Johnson [14] is not immediately applicable because it assumes a more powerful rank operator that retrieves the element with a *precise* rank in L_i . We extend their algorithm to obtain:

Lemma 6. *The approximate union-rank problem can be solved in $O(m(\text{cost}_{\max} + \text{cost}_{\text{rank}}))$ time.*

3.4 Top- k range reporting for $k = O(\text{polylg } n)$

We now give a deterministic structure for approximate range k -selection with $k \leq l = O(\text{polylg } n)$. We discuss only $\lg \lg n \geq \sqrt{B}$ in this subsection, and leave the opposite case to the appendix (see the proof of Theorem 3).

The base tree of our structure is a weight-balanced B-tree [6] T on the points of the input set S . Each internal node of T has at most $f = \lg^\epsilon n$ child nodes where $\epsilon < 1$ is a constant. Each leaf node stores at most $b = flB$ elements. Given a node u , denote by S_u the set of elements stored in the subtree of u , and by G_u the c^2l elements in S_u with the highest scores, where c is the constant c in Lemma 3. For each leaf node z , maintain a structure of [27] for approximate range k -selection on S_z . Consider an internal node u with child nodes u_1, \dots, u_f , where the subscripts reflect the ordering of the elements in their subtrees. We maintain a (f, c^2l) -structure for solving the (f, c^2l) -problem on the (f, c^2l) -group $G_u = (G_{u_1}, \dots, G_{u_f})$. Finally, store the elements of $G_{u_1} \cup \dots \cup G_{u_f}$ in a B-tree so that, given any $1 \leq \alpha_1 \leq \alpha_2 \leq f$, we can find the maximum score of the elements in $\bigcup_{i \in [\alpha_1, \alpha_2]} G_{u_i}$ using $O(\lg_B(fl))$ I/Os.

Given a top- k query with range $q = [x_1, x_2]$, search T in a standard way to obtain $m = O(\lg_f n)$ *canonical subsets* that form a partition of $S \cap q$. Specifically, a canonical subset is either $q \cap S_z$ for some leaf node z , or unions the elements in the subtrees of several continuous child nodes of an internal node u – denote the union as L_u , and let A be the set of all such u . We perform approximate rank k -selection on $\bigcup_{u \in A} L_u$ using Lemma 6: notice that the (f, c^2l) -structure of u and the B-tree on G_u allow us to implement the rank- and max-operators in $O(\lg_B(fl))$ I/Os, respectively (see Lemma 5). Therefore, the approximate rank k -selection finishes in $O(m \lg_B(fl)) = O(\lg_f n \cdot \lg_B f) = O(\lg_B n)$ I/Os; let e be the element returned. For each leaf node z such that $q \cap S_z$ is a canonical subset, perform approximate range k -selection on S_z using q in $O(\lg_B b) = O(\lg_B n)$ I/Os. There are at most two such leaf nodes; let e_1, e_2 be the results of the approximate range k -selection on them, respectively. We return $\max\{e, e_1, e_2\}$ as the final answer.

Moving the other standard details to the appendix, we conclude with our last main result:

Theorem 3. *For top- k range reporting, there is a structure that uses $O(n/B)$ space in the worst case, correctly answers every query with $O(\lg_B n + k/B)$ I/Os in expectation and w.h.p., and can be updated in $O(\lg_B n)$ I/Os in expectation and w.h.p. For $k = O(\text{polylg } n)$, the query and update complexities hold in the worst case.*

Appendix 1: Chernoff bounds

Let X_1, \dots, X_n be independent Bernoulli variables such that $\Pr[X_i = 1] = p_i$. Let $X = \sum_{i=1}^n X_i$ and $\mu = \mathbf{E}[X] = \sum_{i=1}^n p_i$. It holds that:

- for any $\alpha \in (0, 1)$:

$$\Pr[X \geq (1 + \alpha)\mu] \leq e^{-\alpha^2 \mu / 3} \quad (1)$$

$$\Pr[X \leq (1 - \alpha)\mu] \leq e^{-\alpha^2 \mu / 3} \quad (2)$$

- for any $\alpha \geq 2$:

$$\Pr[X \geq \alpha\mu] \leq e^{-\alpha\mu/6} \quad (3)$$

- for any $\alpha \geq 6\mu$:

$$\Pr[X \geq \alpha] \leq 2^{-\alpha}. \quad (4)$$

The above inequalities can be found in many papers and textbooks, e.g., [16, 24].

Appendix 2: Proof of Lemma 1

First, we bound the probability that $|R| \leq 2kp$. For each $1 \leq i \leq |L|$, let x_i be 1 if the i -th element of L is sampled, or 0 otherwise. Let $X = \sum_{i=1}^{|L|} x_i$. Then, $\mathbf{E}[X] \geq 4kp$. Hence:

$$\begin{aligned} \Pr[X \leq 2kp] &= \Pr[X \leq (1/2) \cdot 4kp] \\ &\leq \Pr[X \leq (1/2) \cdot \mathbf{E}[X]] \\ (\text{by Chernoff bound (2)}) &\leq \exp(-\mathbf{E}[X]/12). \end{aligned}$$

To make the above at most $\delta/3$, we need $\mathbf{E}[X] \geq 12 \ln(3/\delta)$, for which it suffices to have $kp \geq 3 \ln(3/\delta)$.

From now on, let e be the element with rank $\lceil 2kp \rceil$ in R , and \hat{k} be the rank of e in L . Consider the event $\hat{k} > 4k$. For each $i \in [1, 4k]$, let y_i be an indicator variable which equals 1 if the i -th greatest element in L is sampled, or 0 otherwise. Let $Y = \sum_{i=1}^{4k} y_i$, i.e., Y the number of samples from the greatest $4k$ elements in L . Clearly, $\mathbf{E}[Y] = 4kp$. Event $\hat{k} > 4k$ implies $Y \leq \lceil 2kp \rceil - 1$. We thus have:

$$\begin{aligned} \Pr[\hat{k} > 4k] &\leq \Pr[Y \leq \lceil 2kp \rceil - 1] \\ (\text{as } Y \text{ is an integer}) &= \Pr[Y < 2kp] \\ &\leq \Pr[Y \leq (1/2) \cdot \mathbf{E}[Y]] \\ (\text{by Chernoff bound (2)}) &\leq \exp(-\mathbf{E}[Y]/12) \end{aligned}$$

To make the above at most $\delta/3$, it suffices to have $kp \geq 3 \ln(3/\delta)$.

Next, we consider the event $\hat{k} < k$. For each $i \in [1, k]$, let z_i be an indicator variable that is 1 if the i -th greatest element in L is sampled, or 0 otherwise. Define $Z = \sum_{i=1}^k z_i$. Thus, $\mathbf{E}[Z] = kp$. Event $\hat{k} < k$ implies $Z \geq \lceil 2kp \rceil$. Therefore:

$$\begin{aligned} \Pr[\hat{k} < k] &\leq \Pr[Z \geq 2kp] \\ &= \Pr[Z \geq 2 \cdot \mathbf{E}[Z]] \\ (\text{by Chernoff bound (3)}) &\leq \exp(-\mathbf{E}[Z]/3) \end{aligned}$$

To make the above at most $\delta/3$, we need $\mathbf{E}[Z] \geq 3 \ln(3/\delta)$. It suffices to make $kp \geq 3 \ln(3/\delta)$.

Appendix 3: Completing the proof of Theorem 1

It remains to analyze the update cost. We focus on the top-constant-score structures because the update time of the counting and τ -reporting structures is straightforward. Recall that there is a top-constant-score structure on each sample set S_i ($0 \leq i \leq \lfloor \lg n \rfloor$). Let X_i be the number of insertions performed on S_i . Set $X = \sum_{i=0}^{\lfloor \lg n \rfloor} X_i$. We will show that $X = O(n)$ with probability at least $1 - 1/n^2$. This implies $\mathbb{E}[X] = O(n)$. As there cannot be more deletions on each X_i than insertions, this will complete the proof.

Set $c = 3 \ln 9$. It suffices to consider only i satisfying $2^i > 3 \ln 9$. X_i is the sum of at least $n/2$ but at most n independent Bernoulli variables each of which equals 1 with probability $c/2^i$. Set $\mu_i = \mathbb{E}[X_i] \in [cn/2^{i+1}, cn/2^i]$. Note that $\mu_i > 1$ for all $i \in [0, \lfloor \lg n \rfloor]$.

For $i \leq \lg(cn) - \lg(9 \ln n) - 1$, by Chernoff bound (3), $\Pr[X_i \geq 2\mu_i] \leq \exp(-\mu_i/3)$, which can be easily verified to be at most $1/n^3$. Now, consider $i > \lg(cn) - \lg(9 \ln n) - 1$. Note that there are $O(\lg \lg n)$ such i . Furthermore, for every such i , it holds that $2^{i+1} > cn/(9 \ln n)$, which means $\mu_i = O(\lg n)$. By Chernoff bound (4), when $n \geq 4$, $\Pr[X_i \geq (\lg n^3)\mu_i] \leq 2^{-(\lg n^3)\mu_i} < 2^{-\lg n^3} = 1/n^3$. In other words, with probability at least $1 - O(\lg \lg n/n^3)$, the sum of X_i of all $i > \lg(cn) - \lg(9 \ln n) - 1$ is at most $O(\lg \lg n \cdot \lg^2 n)$.

The above discussion shows that, with probability at least $1 - 1/n^2$, $X \leq (\sum_{i=0}^{\lfloor \lg n \rfloor} 2\mu_i) + O(\lg \lg n \cdot \lg^2 n) = O(n)$.

Appendix 4: Proof of Corollary 1

Top- k range reporting. Both count and max-score queries can be supported by a slightly augmented B-tree, whereas a τ -reporting query can be answered using the external priority search tree of Arge, Samoladas and Vitter [5]. All structures use linear space, answer a query in logarithmic I/Os (plus linear output cost), and can be updated in logarithmic I/Os per insertion and deletion. Using the extension explained in Section 2, the max-score B-tree also answers a top-constant-search query in logarithmic I/Os.

Top- k stabbing. Count queries can again be supported with an augmented B-tree (e.g., [29]). A max-score structure was given by Agarwal et al. [3], and (using the extension in Section 2) can be modified to handle a top-constant-search query. A τ -reporting query can be answered by the structure of Tao [28]. All structures use linear space, answer a query in logarithmic I/Os (with linear output cost), and can be updated in logarithmic amortized I/Os.

Top- k 3-sided range search. A count query can be answered by the linear-space structure of Govindarajan, Agarwal and Arge [15] in logarithmic I/Os. A max-score query can be answered by the linear-space structure of Sheng and Tao [26] in logarithmic I/Os. Their structure can be easily extended to support top-constant-score queries with the same performance guarantees. Finally, τ -reporting is essentially the $Q(3, 1)$ problem defined by Afshani, Arge and Larsen [1]. They gave a structure with space $O(\frac{n}{B} \frac{\lg n}{\lg_B n})$ and $O(\lg_B n + t/B)$ query time.

The query and update cost in Corollary 1 immediately follows from Theorem 1. Regarding the space usage, it suffices to show that if S has n elements, the total size of all its sample sets maintained in our reduction is $O(n)$ with probability at least $1 - 1/n^2$. This can be proved following an argument similar to the proof of Theorem 1 on the update cost.

Appendix 5: Proof of Theorem 2

It is well-known [18] that, for the *4-sided orthogonal range search* problem in 2d space, a deterministic structure needs $\Omega(\frac{n}{B} \frac{\lg n}{\lg \lg_B n})$ space in the worst case to ensure $O(\lg_B^c n + k/B)$ query time, for any constant c . In 3d space, the $Q(3, 1)$ problem (as defined in [1]) is more general than 4-sided orthogonal range search, whose lower bound thus also applies to the $Q(3, 1)$ problem. Next, we show that if a deterministic structure using $o(\frac{n}{B} \frac{\lg n}{\lg \lg_B n})$ space could answer a top- k 3-sided range query in $O(\lg_B^c n + k/B)$ I/Os, the same structure would also settle a $Q(3, 1)$ query with $O(\lg_B^c n + k/B)$ I/Os, thus reaching a contradiction.

In the $Q(3, 1)$ problem, the input is a set S of n points in \mathbb{R}^3 . Given a rectangle $q = [x_1, x_2] \times [y, \infty) \times [z, \infty)$, a query reports all the points in $S \cap q$. Suppose that we had an efficient structure using $o(\frac{n}{B} \frac{\lg n}{\lg \lg_B n})$ space as mentioned above. We use it to index S by treating the z -coordinate of each point as its score. Then, we answer a $Q(3, 1)$ query by a series of top- k' queries with doubling k' . First, issue a top- k' query with $k' = B \lg_B^c n$ and a 3-sided rectangle $[x_1, x_2] \times [y, \infty)$. Let S' be the set of retrieved points. If the minimum score of the points in S' is below z , we have found all the points satisfying the $Q(3, 1)$ query, and therefore, terminate the algorithm. Otherwise, we double k' and repeat.

The cost of the i -th top- k' query is $O(\lg_B^c n + 2^{i-1} \lg_B^c n) = O(2^{i-1} \lg_B^c n)$. Hence, the cost of all these queries is dominated by that of the last one. It is easy to see that the last top- k' query has cost $O(\lg_B^c n + 2k/B) = O(\lg_B^c n + k/B)$.

Appendix 6: Proof of Lemma 2

The space bound is obvious. If there are n updates, by Chernoff bound (3), the number of insertions on R is at most $2n/(\lg_B n)$ with probability at least $1 - 1/n^2$ when n is larger than 256. There can be at most the same number of deletions on R . Hence, with probability at least $1 - 1/n^2$ the number of updates on R is $O(n/\lg_B n)$; as each update takes $O(\lg_B^2 n)$ amortized I/Os, the total update cost is $O(n \lg_B n)$.

Now we analyze the query algorithm, considering $|S(q)| \geq 4k$ (the opposite case is already taken care of at the beginning of Section 3). For $k \geq 7 \ln n \cdot \lg_B n$, $kp \geq 7 \ln n$ which is at least $3 \ln(3n^2)$ for $n \geq 256$. Applying Lemma 1 with $\delta = 1/n^2$, we know that with probability at least $1 - 1/n^2$: (i) R has $2kp$ elements, and (ii) the element with rank $\lceil 2kp \rceil$ in R has rank between k and $4k$. When both conditions hold, our algorithm reports the top- k elements in $O(\lg_B n + k/B)$ I/Os. Hence, with probability at least $1 - 1/n^2$, our algorithm terminates in $O(\lg_B n + k/B)$ I/Os. With probability at most $1/n^2$ (i.e., at least one of (i) and (ii) is violated), our algorithm spends $O(n/B)$ I/Os answering the query, adding only $\frac{1}{n^2} \cdot O(n/B) = o(1)$ to the expected cost.

Appendix 7: Proof of Lemma 4

We prove the lemma also with tabulation. Let us define the list of the $B \lg_B(fl)$ largest elements of G_i ($1 \leq i \leq f$) as the *prefix* of G_i , and denote it as P_i . Let \mathbf{P} be the union of P_1, \dots, P_f ; we refer to \mathbf{P} as a *prefix set*. \mathbf{P} contains at most $fB \lg_B(fl)$ points. We compress \mathbf{P} by describing each element e (say, $e \in G_i$ for some i) in \mathbf{P} using its global rank in G and its local rank in G_i , for which purpose $O(\lg(fl))$ bits suffice. Hence, \mathbf{P} can be described by $O(fB \lg_B(fl) \cdot \lg(fl)) = O(\lg^\epsilon n \cdot \lg \lg n \cdot \lg_B \lg n \cdot \lg \lg n) = O(\lg^\epsilon n \cdot (\lg \lg n)^3)$ bits, which fits in a word.

We ensure that a compressed \mathbf{P} is always described by the same number of bits – for this purpose, we force each prefix to have $B \lg_B(fl)$ elements by appending dummy elements as needed. In a compressed \mathbf{P} , the elements are first sorted by which sets they come from, and then by their local ranks. In this way, the

global rank of the element with local rank $r \leq B \lg_B fl$ in G_i can be found in constant time for any r and i . Let λ be the number of possible compressed prefix sets, i.e., $\lambda = 2^{O(\lg^\epsilon n \cdot (\lg \lg n)^3)}$.

Suppose that we need to delete an element e from G_i . If $e \in P_i$, using a B-tree on G , we find the global rank r of e in $O(\lg_B(fl))$ I/Os. We observe that the new compressed prefix set \mathbf{P}' is determined by the current compressed prefix set \mathbf{P} , i and r . To see this, first consider a compressed prefix $P_{i'}$ with $i' \neq i$: if an element has global rank at least r , it should have its global rank decreased by 1. Regarding the compressed prefix P_i , the same is true; furthermore, all such elements in P_i should also have their local ranks decreased by 1. Finally, the last element of P_i is discarded (i.e., marked dummy).

Motivated by the above observation, we precompute a table that contains an entry for every combination of (\mathbf{P}, i, r) , and stores in that entry the corresponding \mathbf{P}' . The entries are sorted in the lexicographic order of the bit description of (\mathbf{P}, i, r) to allow constant-time lookups (by that description). The number of entries is $\lambda \cdot f \cdot B \lg_B(fl) = O(\lambda \cdot \text{poly} \lg n)$. Hence, the whole table contains $O(\lambda \cdot \text{poly} \lg n \cdot \lg^\epsilon n \cdot (\lg \lg n)^3) = o(n)$ bits. Furthermore, as the \mathbf{P}' in each cell can be easily computed in $O(\text{poly} \lg n)$ time, the table can be generated in $O(\lambda \cdot \text{poly} \lg n) = o(n)$ time.

Finally, we need to insert into P_i the element with score rank $B \lg_B fl$ in G_i . Following the above ideas, in general, an insertion can also be handled in $O(\lg_B(fl))$ I/Os by pre-computing another lookup table occupying $o(n)$ bits. This table can also be generated in $o(n)$ time.

Excluding the pre-computed tables, the structure includes: (i) a B-tree on each G_i , (ii) a B-tree on G , and (iii) a compressed prefix set which fits in one word. The space is therefore linear. All the B-trees can be easily built in $O(fl \cdot \lg_B(fl))$ I/Os. By querying the B-tree on G , we can easily generate each P_i (including the local and global ranks of the elements) in $O(fl \cdot \lg_B(fl))$ I/Os, after which the compressed prefix set can be obtained in $O(fl/B)$ I/Os.

Appendix 8: Proof of Lemma 5

We now explain how to handle a deletion. Suppose that an element e_{old} is to be deleted in G_i for some $i \in [1, f]$. Let r_{old} be the rank of e_{old} in G before the deletion. Except possibly for only one pivot, the new compressed sketch set can be deduced based only on the current compressed sketch set, r , i and the value of $|G_i|$ before the deletion. To see this, consider first $\Sigma_{i'}$ where $i' \neq i$. Each pivot whose global rank is larger than r_{old} now needs to have its global rank decreased by 1. Regarding Σ_i , the same is true, and every such pivot should also have its local rank decreased by 1. Furthermore, the last pivot of Σ_i should be discarded (i.e., marked dummy) if $|G_i|$ was a power of 2 before the deletion: in such a case, we say that Σ_i *shrinks*. Finally, if e_{old} happens to be a pivot of Σ_i (but not the last one), then a new pivot needs to be computed to replace it – this is the only pivot that cannot be deduced; we call it a *dangling* pivot.

We precompute a *deletion lookup table* \mathbb{T}_{del} . Let Σ denote a possible compressed sketch set. \mathbb{T}_{del} contains an entry for every possible combination of $(\Sigma, r_{old}, i, |G_i|)$, where the meanings of r_{old} , i and $|G_i|$ are as explained above. This entry contains the new compressed sketch set determined by $(\Sigma, r_{old}, i, |G_i|)$ – in case there is a dangling pivot in the new sketch set, set its global rank with a special mark (which can be done by using one extra bit to represent each global rank). We sort the entries in the lexicographic order of the bit-strings of the corresponding $(\Sigma, r_{old}, i, |G_i|)$ to allow constant-time lookups. Just like \mathbb{T}_{ins} , an entry in \mathbb{T}_{del} is associated with a pointer to a linked list indicating the invalidated pivots of the sketch set stored in that entry. \mathbb{T}_{del} occupies the same space as \mathbb{T}_{ins} , and can be constructed with the same cost.

The concrete steps of deleting e_{old} are as follows. After obtaining its global rank r_{old} in $O(\lg_B(fl))$ I/Os, we look up \mathbb{T}_{del} to find the new compressed sketch set. If Σ_i shrinks, we delete the last pivot from the B-tree on (the uncompressed) Σ_i in $O(\lg_B \lg l)$ I/Os. If e_{old} was a pivot (say, the j -th one for some j),

we retrieve the element e with local rank $\lfloor \frac{3}{2} \cdot 2^{j-1} \rfloor$ in G_i , and obtain its global rank r . This can be done using $O(\lg_B(fl))$ I/Os in total. We then use e to replace $\Sigma_i[j]$ in $O(\lg_B \lg l)$ I/Os. Finally, re-compute the invalidated pivots (if any) in the same way as in insertion. As analyzed in Section 3.2, such re-computation increases the update cost by $O(\lg_B(fl))$ amortized I/Os.

Excluding the pre-computed tables, the structure includes: (i) a B-tree on G , (ii) a B-tree on each G_i , (iii) a B-tree on each Σ_i , and (iv) a compressed sketch set which fits in one word. The space is therefore linear. All the B-trees can be easily built in $O(fl \cdot \lg_B(fl))$ I/Os, after which the compressed prefix set can be obtained in $O(fl/B)$ I/Os.

Appendix 9: Proof of Lemma 6

Case $k \geq m$. We will collect a set P of *pivots*, each of which is an element in some set L_i . The pivot collection is carried out in $\lceil \lg_c m \rceil$ rounds. In the j -th round ($1 \leq j \leq \lceil \lg_c m \rceil$), $\lceil m/c^{j-1} \rceil$ sets of L_1, \dots, L_m are *active*, while the other sets are *inactive*. At the beginning, all of L_1, \dots, L_m are active.

In round $j \in [1, \lceil \lg_c m \rceil]$, we use the rank-operator to request an element with rank $c^j k/m$ in each of the active sets. Remember that the operator can return any element whose rank in an active set is at least $c^j k/m$ and less than $c^{j+1} k/m$. Let P' be the set of elements fetched. Call each element in P' a *marker*, and assign it a *weight* $\lceil c^j k/m \rceil - \lceil c^{j-1} k/m \rceil = \Theta(c^j k/m)$. Pick the $\lceil m/c^j \rceil$ largest markers in P' as pivots, and add them to P , among which the smallest is the *cutoff pivot* of this round. An active set remains active if its marker was added to P in this round, whereas the other active sets become inactive.

After $\lceil \lg_c m \rceil$ rounds, P has $\sum_{j=1}^{\lceil \lg_c m \rceil} \lceil m/c^j \rceil = O(m)$ pivots. We perform a *weighted selection* to find the largest element $e \in P$, such that the total weight of all pivots greater than or equal to e is at least k . The algorithm terminates by returning e .

Analysis. The algorithm finishes in $O(m \cdot \text{cost}_{\text{rank}})$ times is because the j -round takes $O(m/c^{j-1})$ time (i.e., geometrically decreasing with j), and the weighted selection of finding e from P takes $O(m/B)$ time. Next, we show that the rank of e in L falls in the range $[k, c'k]$ for some constant c' .

In any S_i ($1 \leq i \leq m$), the pivot taken a round must rank behind all those of earlier rounds, because the pivot of the j -th round has rank in $[c^j k/m, c^{j+1} k/m)$. Furthermore, the cutoff pivot of each round cannot be larger than e , because at each round j , we ensure that at least $\lceil m/c^j \rceil (c^j k/m) \geq k$ elements in L are at least the cutoff pivot. Each S_i has at least one marker smaller than or equal to e – refer to the largest such marker the *succeeding marker* of S_i . Note that the marker may not necessarily be a pivot.

The rest of the proof is similar to the one in [14]. We sketch it here for completeness. If S_i has at least one pivot at least e , let p_i be the smallest such pivot, and call S_i a *pivotal set*. Let $S_i[e, p_i)$ be the set of elements in S_i that are at least e but smaller than p_i , and $S_i[p_i, \infty)$ be the set of elements in S_i that are at least p_i . The size of $S_i[e, p_i)$ is asymptotically no greater than that of $S_i[p_i, \infty)$. This is because (i) the elements in $S_i[e, p_i)$ must fall between p_i and the succeeding marker of S_i , (ii) hence, $|S_i[e, p_i)|$ is at most $O(1)$ more than the weight of the succeeding marker, and (iii) the weight of the succeeding marker is $O(|S_i[p_i, \infty)|)$. Hence, in the pivot sets, the total number of elements at least e is $O(k)$. On the other hand, each non-pivotal set S_i has less than $c^2 k/m$ elements at least e . They altogether contribute less than $c^2 k = O(k)$ such elements.

Case $k < m$. From each S_i , we use the max-operator to request the max element in S_i . Let P' be the set of elements fetched (i.e., one from each S_i). We identify the k largest elements in P' ; let e' be the k -th largest element in P' . We turn a set S_i inactive if its max element is smaller than e' . Run the above $k \geq m$

algorithm on the k remaining active sets. Suppose that the algorithm returns e . We return $\max\{e, e'\}$ as the final answer. It is easy to prove that the algorithm is correct, and runs in time $O(m(\text{cost}_{\text{rank}} + \text{cost}_{\text{max}}))$.

Appendix 10: Proof of Theorem 3

We now complete the description in Section 3.4 of the structure for $k = O(\text{poly} \lg n)$. This structure uses linear space, answers a query in $O(\lg_B n + k/B)$ space, and can be updated in $O(\lg_B n)$ I/Os. This, together with Lemma 2, establishes the theorem.

$\lg \lg n \geq \sqrt{B}$. Let us continue the discussion of Section 3.4. To support updates, for each internal node u , build another B-tree on the scores of the elements in G_u . Similarly, for each leaf node z , build a B-tree on the scores of the elements in S_z . Refer to these B-trees as *score B-trees*.

To insert a point e in S , we first descend a root-to-leaf path π to the leaf node z where e should be stored. Remember that π has $O(\lg_f n)$ nodes. At z , update all its secondary structures in $O(\lg_B^2 b) = O((\lg_B \lg n)^2) = O(\lg_B n)$ I/Os. Next, we fix the secondary structures of the nodes along π in bottom-up order. Let $\text{parent}(z)$ be the parent node of z . If e enters G_z , at $\text{parent}(z)$, delete the element in G_z with the lowest score, and insert e in G_z . Accordingly, the secondary structures of $\text{parent}(z)$ are updated. In general, after updating an internal node u , we check using the score B-tree of u whether e enters G_u . If yes, at $\text{parent}(u)$, delete the element in G_u with the lowest score, insert e in G_u , and update the secondary structures of $\text{parent}(u)$. By Lemma 5, we spend $O(\lg_B(fl))$ I/Os at each node, and hence, $O(\lg_B n)$ I/Os in total along the whole π .

We now describe how to handle node splits. Suppose that a leaf node z splits into z_1, z_2 . First, build the secondary structures of z_1 and z_2 in $O(b \lg_B^2 b)$ I/Os. At $v = \text{parent}(z)$, destroy G_z , and include G_{z_1} and G_{z_2} into G_v . Rebuild all the secondary structures at v in $O(fl \cdot \lg_B(fl)) = O(b \lg_B b)$ I/Os (Lemma 5). This cost can be amortized over the $\Omega(b)$ updates that must have taken place in z , such that each update is charged only $O(\lg_B^2 b) = O(\lg_B n)$ I/Os.

A split at an internal level can be handled in a similar way. Suppose that an internal node u splits into u_1, u_2 . Divide G_u into G_{u_1} and G_{u_2} in $O(fl/B)$ I/Os, and then rebuild the secondary structures of u_1, u_2 in $O(fl \cdot \lg_B(fl))$ I/Os. After discarding G_u but including G_{u_1}, G_{u_2} , we rebuild the secondary structures of $\text{parent}(u)$ in $O(fl \cdot \lg_B(fl))$ I/Os. On the other hand, $\Omega(fl)$ updates must have taken place in the subtree of u (recall that the base tree is a weight balanced B-tree). Hence, each of those updates bears $O(\lg_B(fl))$ I/Os for the split cost. As an update bears such cost for at most one node per level, the amortized update cost increases by only $O(\lg_B n)$.

An analogous algorithm can be used to handle a deletion in $O(\lg_B n)$ amortized I/Os. The details should have become straightforward.

Regarding space consumption, there are $O(n/(fb))$ internal nodes, each of which occupies $O(fl/B)$ blocks. Hence, all the internal nodes use $O(\frac{n}{fb} \cdot \frac{fl}{B}) = O(\frac{n}{B^2})$ space in total. The overall space cost is therefore $O(n/B)$. Global rebuilding is performed to ensure that the precomputed lookup tables always consume $O(n/B)$ space. Specifically, after n has changed by a factor of 2, we destroy the entire structure, and rebuild everything (including the lookup tables) in $O(n \lg_B n)$ I/Os, making sure that the lookup tables can be used until the input size increases to $2n$.

$\lg \lg n \leq \sqrt{B}$. In this case, the base tree T is a weight balanced B-tree on the points of S where an internal node has at most $f = \sqrt{B}$ child nodes, and a leaf node stores at most $b = flB$ elements. For each node u of T , define S_u and G_u as before. At each leaf node z , store S_z in a structure of [27] for approximate range k -selection. Consider an internal node u with child nodes u_1, \dots, u_f (ordered in the same way as explained

in Section 3.4). For each G_{u_i} , create a logarithmic sketch Σ_{u_i} , which has $O(\lg l) = O(\lg \lg n)$ pivots. Since $\lg \lg n \leq \sqrt{B}$, all the $\Sigma_{u_1}, \dots, \Sigma_{u_f}$ can be stored in $O(1)$ blocks. Use another block to store the element with the maximum score in each G_{u_i} .

The height of the tree is $O(\lg_B n)$. A top- k range query is processed in the same manner as explained in Section 3.4 except that, when applying Lemma 6, both the max- and rank-operators at each internal node can now be implemented in $O(1)$ I/Os. The query cost is therefore $O(\lg_B n)$.

The structure clearly occupies linear space. It can be made dynamic using the strategies illustrated earlier for the $\lg \lg n \geq \sqrt{B}$ case, so that an insertion/deletion is performed in $O(\lg_B n)$ amortized I/Os. We omit the tedious details which should have become straightforward.

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